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A GENERALIZED FINITE ELEMENT METHOD FOR MODELING ARBITRARY INTERFACES IN LARGE DEFORMATION PROBLEMS

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Abstract. In this paper, a generalized–FEM technique is presented in modeling of arbitrary interfaces in large deformations. The method is used to model the internal interfaces and arbitrary geometries using a uniform non-conformal mesh. The technique is applied to capture independent deformations at both sides of separated element cut by the interface in a uniform regular mesh. In this approach, a uniform non-conformal mesh is decomposed into sub-elements that conform to the internal interfaces. The geometry of interface is used to produce various triangular, quadrilateral and pentagonal elements at the intersection of interface with regular FE mesh, in which the extra degrees-of-freedom are defined along the interface. The level set method is employed to describe the material geometry on the background mesh. The technique is used to extrude any arbitrary geometry from an initial background mesh and model under different external effects. The most feature of the technique is to introduce the conformal decomposition finite element method, in which the new conforming elements are produced in the uniform structured mesh by decomposing the uniform mesh into elements that is conformed to the material interfaces. Finally, several numerical examples are analyzed to demonstrate the efficiency of proposed technique in modeling arbitrary interfaces in large deformations.

1 INTRODUCTION

In computational mechanics, modeling the internal interfaces and arbitrary geometries using a non-conformal uniform mesh is of great importance. Adaptive mesh strategy and conforming mesh generation for preserving the mesh to the shape of geometry at various stages of solution may consume high expenses of capacity and time. Thus, it is necessary to perform an innovative procedure to alleviate these difficulties by allowing the internal interfaces and arbitrary geometries to be mesh-independent. In fact, an approach that avoids the remeshing is preferable not only in the cost of creating a new mesh, but the tremendous overhead associated with adapting post-processing techniques, such as time histories of specified points, to sequences of meshes in evolution problems. The major appeal of such technique for incorporating discontinuities in finite elements is that it does not require the mesh to conform to discontinuities in the approximation function, or its derivatives.

There are several approaches proposed by researchers over past few decades to model discontinuity problems based on the mesh-free methods [1–3], the moving mesh technique [4], the incorporation of a discontinuous mode on an element level [5] and etc. Among various techniques, the generalized finite element method (G-FEM) [6, 7] and the extended finite element method (X-FEM) [8, 9] have been successfully employed for the weak and strong discontinuities. The capability of X-FEM has been shown in various problems, including: fracture mechanics problems [10–14], large plastic deformations [15–17], and contact friction problems [18–20]. A technique was proposed by Ventura et al. [21] based on a local enrichment of FE space by closed form solutions for dislocations in infinite media via the local partition of unity. Fries and Belytschko [22] proposed a method for arbitrary discontinuities without additional unknowns. A technique was introduced by Gracie et al. [23] in modeling of multiple dislocations based on interior discontinuities.

The X-FEM technique has been extensively employed to minimize the requirement of mesh generation in the problem with internal interfaces. In this method, the enrichment functions are defined to deal with the discontinuity of displacement inside the enriched element. In fact, the X-FEM method addresses the arbitrary interfaces without generating a boundary-fitted mesh by defining the extra degrees-of-freedom in the elements cut by the interfaces. Additional unknowns may be assigned to the mesh entities, such as: elements, nodes, or edges, by introducing additional equations for these unknowns based on the quadrature techniques for the resulting discontinuous interpolation [24, 25]. In this study, an alternative approach is presented, in which a uniform non-conformal mesh is decomposed into triangular, quadrilateral and pentagonal elements that conform to the internal interfaces and arbitrary geometries. The geometry of interface is used to define the extra degrees-of-freedom by adding nodal points that lie on the interfaces. The technique may be considered as a generalized finite element method introduced by Li et al. [26] using a Cartesian Grid with Added Nodes into the unstructured finite elements. In the FE based Cartesian Grid with Added Nodes method, the added nodes increase the size of the linear system of equations and significantly affect the structure of the matrix, which makes it undesirable compared to other generalized FEM techniques, such as Immersed FE methods. However, in the conformal decomposition finite element method proposed here, the new conforming elements are produced in the uniform structured mesh by decomposing the uniform mesh into elements that is conformed to the material interfaces. This method can be used for the multi-material problems, in which the mesh does not necessary to be conformed to the geometry of the materials. In order to describe the material geometry on the background mesh, the level set method is employed to represent the decomposition of non-conformal elements into the conformal sub-elements [27, 28]. The level set technique is used to extrude any arbitrary geometry from an initial background mesh and model under different external effects.

The construction of conforming finite elements based on polygonal meshes was proposed by Sukumar and Tabarraei [27]. The method provides a great flexibility in mesh generation of solid mechanics problems, which involve a significant change in the domain of material. The trial and test functions of polygonal finite elements have been generally constructed based on the approximation functions of mesh-free methods and computational geometry. A particular and notable contribution is based on the mesh-free, or natural–neighbor, basis functions on a canonical element combined with an affine map to construct conforming approximations on convex polygons. This numerical formulation enables the construction of conforming approximation on any polygons, and hence extends the potential applications of finite elements to convex polygons of arbitrary order [28]. The present study illustrates the presentation of the conformal decomposition finite element method in large deformations, in which the new conforming elements are generated in the uniform structured mesh by decomposing the uniform mesh into sub-elements that is conformed to the material interfaces.

The plan of the paper is as follows; Section 2 is devoted to the concept of conforming polygonal finite elements. The implementation of conforming–FEM technique based on the polygonal elements is demonstrated in section 3. The procedure, in which the new conforming elements are produced in the uniform structured mesh by decomposing the uniform mesh into sub-elements, is described in this section. In section 4, several numerical examples are analyzed to demonstrate the efficiency of proposed technique in modeling arbitrary interfaces in large deformations. Finally, some concluding remarks are given in section 5.

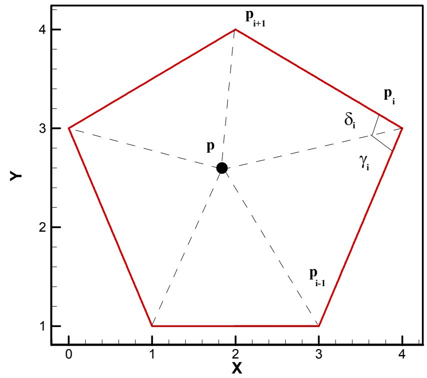


Figure 1. :A pentagon element; Element definition

2 CONFORMING POLYGONAL FEM

The construction of barycentric co-ordinates and the evaluation of shape functions on irregular polygons were originally proposed by Wachspress [29] based on the rational basis functions on polygonal elements. Wachspress [29] employed the principles of perspective geometry [30] to validate the nodal interpolation and linearity on the boundaries. Various aspects of the Wachspress basis function were presented in literature, including: the implementation in numerical analysis by Gout [31], the generalization to convex polytopes by Warren [32], the implementation to construct surface patches by Dahmen et al. [33], etc. The Wachspress basis function was employed into the finite element method by Dasgupta and Malsch [34, 35] to construct the shape functions for concave elements. Rashid and Gullett [36] proposed the technique to construct the shape functions for convex and non-convex elements using a constrained minimization procedure. The construction of conforming finite elements based on polygonal meshes was performed by Sukumar and Tabarraei [27].

An expression for the Wachspress shape functions was given by Meyer et al. [37] as

$$\varphi_i^w(\boldsymbol{x}) = \frac{w_i(\boldsymbol{x})}{\sum_{j=1}^{5} w_j(\boldsymbol{x})} \tag{1}$$

Where

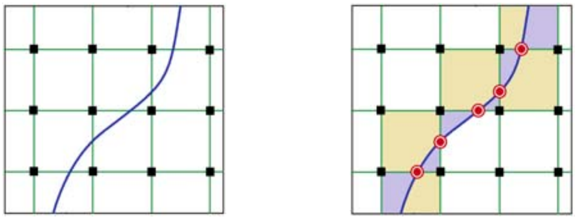


Figure 2. Decomposition of non-conformal elements cut by the interface into conformal sub-elements; ■ Original nodal points, New degrees-of-freedom along the interface

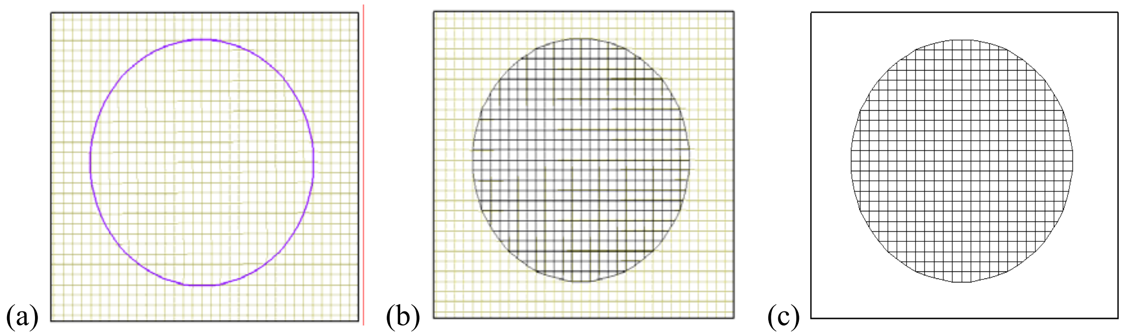


Figure 3. a) Definition of an interface in the uniform non-conformal mesh, b) Determination of standard elements and conformal sub-elements within the material zone, c) Elimination of elements not within the material zone

$$w_i(\boldsymbol{x}) = \frac{A(p_{i-1}, p_i, p_{i+1})}{A(p_{i-1}, p_i, p)\, A(p_i, p_{i+1}, p)} = \frac{\cot\gamma_i + \cot\delta_i}{\|\boldsymbol{x} - \boldsymbol{x}_i\|^2} \tag{2}$$

where $A(a,b,c)$ is the signed area of triangle $[a,b,c]$, and γ_i and δ_i are shown in Figure 1. Since $\cot\gamma_i + \cot\delta_i \equiv \sin(\gamma_i + \delta_i)/(\sin\gamma_i \sin\delta_i)$, the shape functions $\varphi_i^w(\boldsymbol{x})$ have non-negative values and the polygon must be convex, i.e. $\gamma_i + \delta_i < \pi$. The evaluation of the Wachspress basis function can be carried out using the elementary vector calculus operations, as demonstrated by Meyer et al. [37]. Considering the coordinates of the vertices of triangle (p_i, p_{i+1}, p) as (a_1, a_2), (b_1, b_2) and (x_1, x_2), respectively, the value of $\cot\delta_i$ can be computed by

$$\cot\delta_i = \frac{(\mathbf{p}_{i+1} - \mathbf{p}_i).(\mathbf{p} - \mathbf{p}_i)}{|(\mathbf{p}_{i+1} - \mathbf{p}_i)\times(\mathbf{p} - \mathbf{p}_i)|} = \frac{(b_1 - a_1)(x_1 - a_1) + (b_2 - a_2)(x_2 - a_2)}{(b_1 - a_1)(x_2 - a_2) - (x_1 - a_1)(b_2 - a_2)} \equiv \frac{C}{S} \tag{3}$$

and its derivatives can be evaluated as

$$\begin{aligned} \frac{\partial(\cot\delta_i)}{\partial x_1} &= \frac{(b_1 - a_1) - \cot\delta_i\,(a_2 - b_2)}{S} \\ \frac{\partial(\cot\delta_i)}{\partial x_2} &= \frac{(b_2 - a_2) - \cot\delta_i\,(b_1 - a_1)}{S} \end{aligned} \tag{4}$$

The value of $\cot\gamma_i$ and its derivatives can be computed in a similar manner. Finally, the Wachspress shape function $\varphi_i^w(\boldsymbol{x})$ can be obtained according to relation (1).

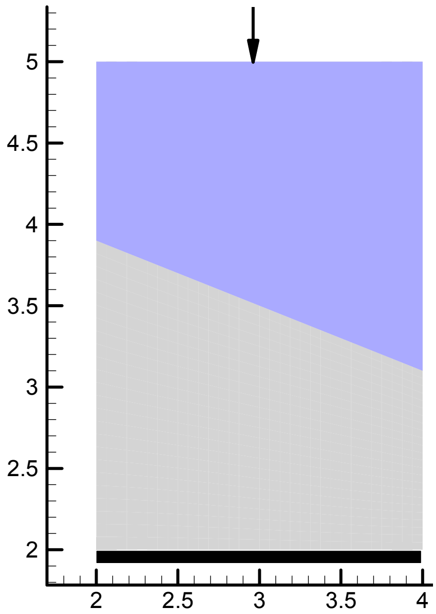


Figure 4: problem view for error analysis

3 GENERALIZED FEM WITH POLYGONAL ELEMENTS

In order to model arbitrary interfaces in a uniform mesh, the regular non-conformal mesh is decomposed into sub-elements that conform to the internal interfaces. In this approach, the concept of conformal decomposition finite element method is used to produce the conforming polygonal elements in the uniform structured mesh by decomposition of the uniform mesh into sub-elements that is conformed to the material interfaces. The geometry of interface is used to produce various polygonal elements at the intersection of interface, as shown in Figure 2, in which the extra degrees-of-freedom are defined along the interface. The position of material interface is determined according to the initial uniform mesh by using the level set method.

The level set method is employed to describe the material interface by extruding arbitrary geometry from the initial background mesh. The technique is used to represent the geometry of interface on the structured, non-conformal mesh. The level set method performs the decomposition of non-conformal elements into conformal sub-elements by introducing the material interface based on the sign of level set function. The performance of this conformal decomposition affects the quality of conformal sub-elements. In general, the conformal decomposition must be robustly handled for unacceptable and degenerate cases. These situations can be occurred whenever the interface passes through a nodal point. In such case, a robust scheme is needed for handling nearly degenerate cases. If nearly degenerate elements are not addressed, the resulting matrix system may be numerically singular.

A general procedure for handling the conformal decomposition can be performed by determination of the edges of non-conformal elements cut by the material interface. An edge is assumed to be cut by the interface if the level set values of two nodal points supported by the edge have different signs. The procedure to handle the nodal points with zero level set values, or nearly zero level set values is optional, but they must be handled consistently. For the edge of element cut by the interface, new degrees-of-freedom are introduced at the edge of non-conformal element, as shown in Figure 2. The coordinates of this point can be obtained by linear interpolation on the edge of element. For an edge with nodal level set values of φ_1 and φ_2, the coordinates of new point can be obtained as

$$\boldsymbol{x}_i = \boldsymbol{x}_1 + \alpha(\boldsymbol{x}_2 - \boldsymbol{x}_1) \tag{5}$$

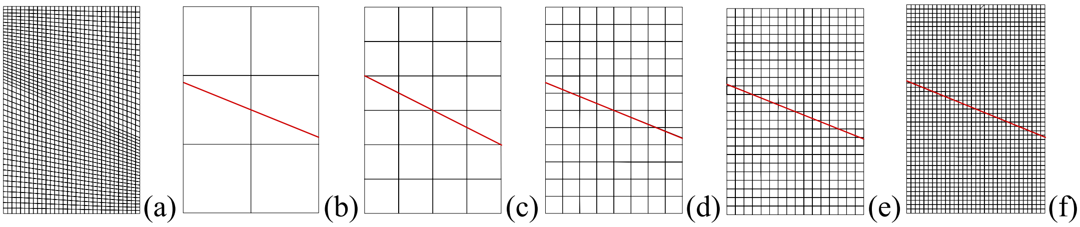


Figure 5: different mesh sizes used for error analysis

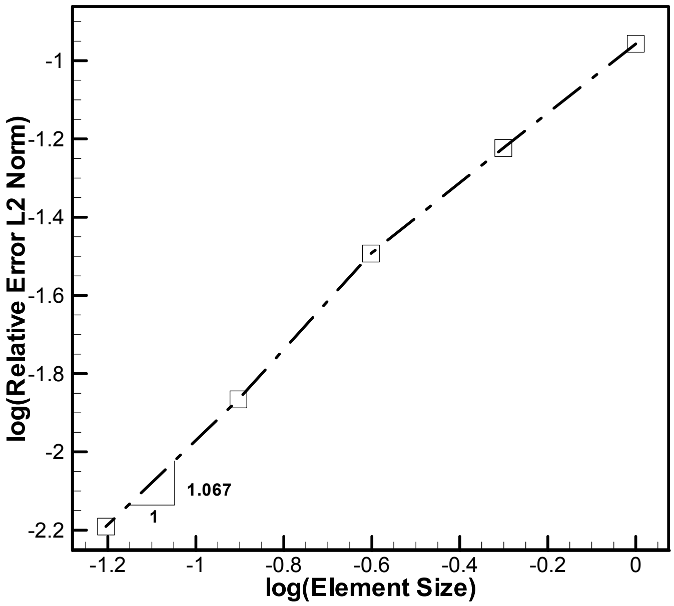


Figure 6: Displacement relative error L2 norm versus Mesh size

where $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are the coordinates of new point and the value of α is defined by a linear interpolation as $\alpha = \|\varphi_1\|/h$,with h denoting the size of element. However, if $\alpha < \varepsilon$, or $\alpha > 1-\varepsilon$, in which the parameter ε is assumed to be 0.05, the new point is not generated. In this case, the interface passes through the nearest nodal point of the edge, and the level set value is set to zero at the nearest node. A detailed study of the sensitivity analysis to this decomposition parameter has not been performed here. However, it is obvious that a large value of parameter ε may cause significant errors due to deficiency between the prescribed geometry and the decomposed geometry. In addition, a small value of parameter ε results in multiple nodal points that is numerically coincident.

For the conformal sub-elements, the new point along the interface is added to the original vertex nodal points. Since the new point may be coincident with the vertex nodal points, different cases can be occurred for the conformal sub-elements.Various polygonal elements can be generated according to the position of interface in the regular uniform mesh, including the triangular, quadrilateral and pentagonal elements. If the interface passes through a nodal point, or nearest nodal point ($\alpha < \varepsilon$ or $\alpha > 1-\varepsilon$), it results in the triangular–quadrilateral sub-elements. If the interface cuts two edges of non-conformal element, the conformal decomposition results in two quadrilateral sub-elements, or the triangular–pentagonal sub-elements. The conformal decomposition strategy of degenerate cases depends on the path of interface across the edge and nodal points of the element. The conformal decomposition may result in two triangular sub-elements with no new degrees-of-freedom. If the interface passes through the edge of an element, or nearest nodal points of the edge, there is no conformal decomposition and no new degrees-of-freedom. By defining the material interface in the uniform non-conformal mesh and performing the conformal decomposition to generate various polygonal sub-elements, the standard elements and conformal sub-elements within the material zone must be first determined; those elements or sub-elements which are not within the material zone must be then removed, as shown in Figure 3, and the generalized finite element model is finally analyzed under the external loading.

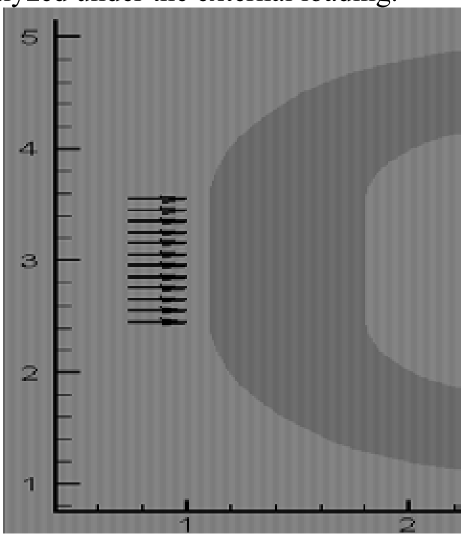


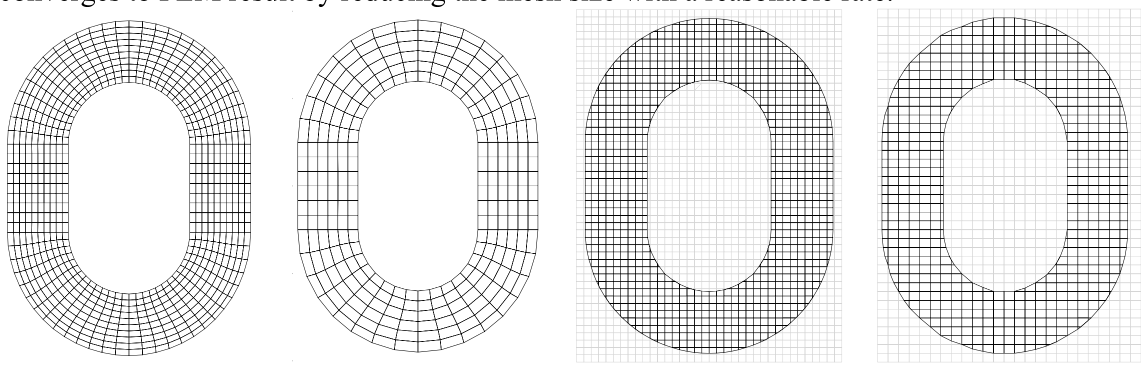
Figure 7. Pressing of an elastic ring; Problem definition

3 NUMERICAL RESULTS

In order to illustrate the accuracy and versatility of the generalized-FEM technique several numerical examples with curve interfaces are presented. The examples are solved using both the G-FEM and FEM techniques, and the results are compared. In order to perform a real comparison, the same number of elements are assumed for both the 'coarse' and 'fine' meshes independent of the shape of discontinuity to assess the accuracy of discretization. All numerical examples are modeled by a plain strain representation and the convergence tolerance is set to 10^{-14}.

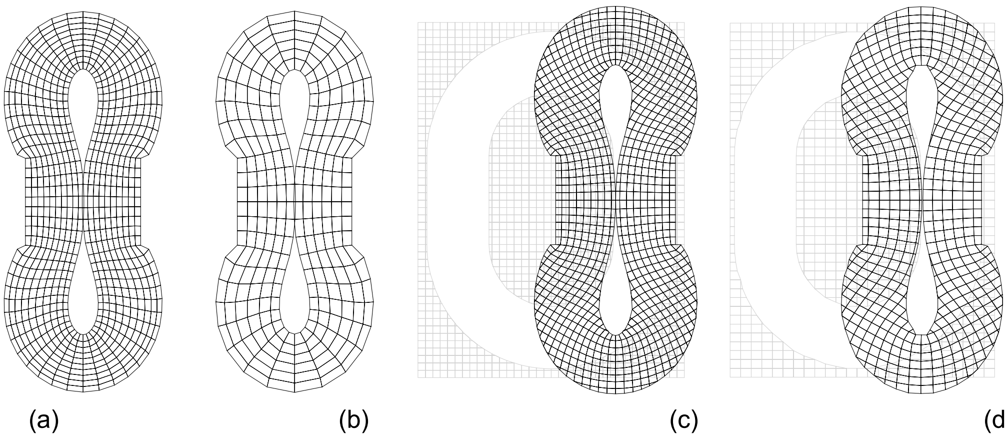
3.1 DISPLACEMENT RELATIVE ERROR L2 NORM ANALYSIS

For determining accuracy of the method, we investigate the accuracy of the problem using error analysis of the method. We consider a simple problem as depicted in Figure 4. The problem is a general rectangular specimen with an inclined internal interface which have two different materials in each side of the interface. We constrained bottom edge of specimen and apply a uniform displacement on the top edge.for error analysis we use a dens standard FEM mesh as reference depicted in Figure 5a. For determining the sensitivity of the GFEM to mesh size we use 5 mesh with different sizes as depicted in Figure 5b to 5f. The result of displacement relative L2 error norm is depicted in figure 6, which shows the GFEM converges to FEM result by reducing the mesh size with a reasonable rate.



(a) (b) (c) (d)

Figure 8. Pressing of an elastic ring; a) FEM mesh of 740 elements, b) FEM mesh of 264 elements, c) Generalized–FEM mesh of 1813 elements, d) Generalized–FEM mesh of 1000 elements



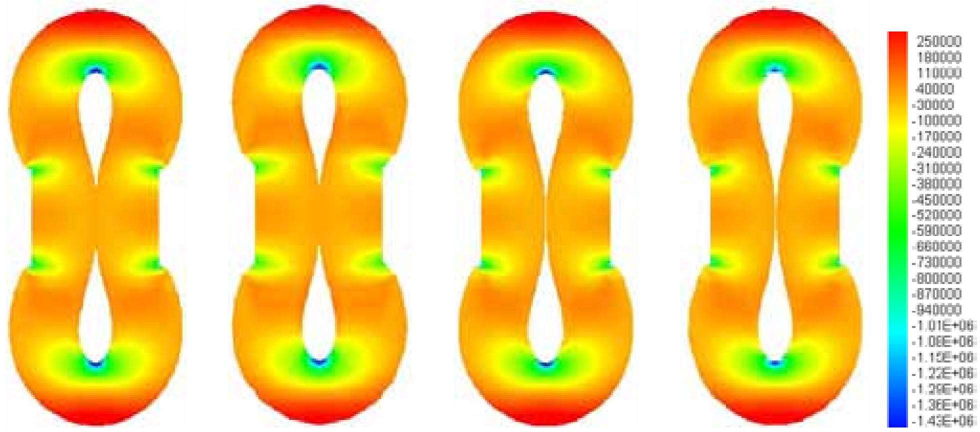
(a) (b) (c) (d)

Figure 9. Deformed configurations at 1.4 cm; a) FEM mesh of 740 elements, b) FEM mesh of 264 elements, c) Generalized–FEM mesh of 1813 elements, d) Generalized–FEM mesh of 1000 elements

3.2 PRESSING OF AN ELASTIC RING

The last example refers to the pressing of an elastic ring, as shown in Figure 7. The ring is restrained at the right edge, and a uniform deformation of 1.48 cm is imposed at the left edge.

The ring is assumed to be elastic with the Young modulus of $2.1\times10^{6}\,\mathrm{Kg/cm^2}$ and Poisson ratio of 0.35. Two conformal meshes of 740 and 264 quadrilateral elements are employed in the FEM analysis and two uniform non-conformal meshes of 1000 and 1813 quadrilateral elements in the G-FEM analysis, as shown in Figure 8. In G-FEM, the non-conformal grid is decomposed into sub-elements, in which the geometry of interface is used to produce various polygonal elements at the intersection of interface together with the extra degrees-of-freedom defined along the interface. As can be observed from Figures 8(c–d), the elements and sub-elements that are not within the material zone are removed, and the G-FEM model is analyzed under the prescribed displacement. The deformed configurations of the G-FEM and FEM models are shown in Figure 9 at the deformation of 1.48 cm. In Figure 10, the distribution of normal stress σ_x contours are presented for both techniques at the final stages of pressing. A good agreement can be seen between the G-FEM and FEM approaches. Finally, a comparison of the reaction force versus vertical displacements is performed between the G-FEM and FEM in Figure 11.



(a) (b) (c) (d)

Figure 10. The distribution of normal stress contours at 1.4 cm; a) FEM mesh of 740 elements, b) FEM mesh of 264 elements, c) Generalized–FEM mesh of 1813 elements, d) Generalized–FEM mesh of 1000 elements

4 CONSLUSION

In the present paper, a generalized–FEM technique was presented in modeling of arbitrary interfaces in large deformation problems. A technique was proposed by conformal decomposition of FEM, in which the new conforming sub-elements were generated in the uniform structured mesh that conform to the internal interfaces. The method was used to model the arbitrary geometries using a uniform non-conformal mesh. The geometry of interface was used to produce various triangular, quadrilateral and pentagonal elements at the intersection of interface with regular FE mesh, in which the extra degrees-of-freedom were defined along the interface. The level set method was employed to describe the material geometry on the background mesh by extruding arbitrary geometry from an initial background mesh. By defining the material interface in the non-conformal mesh and performing the conformal decomposition to generate various polygonal sub-elements, the standard elements and conformal sub-elements within the material zone were determined, and those elements or sub-elements which are not within the material zone were removed. Finally, the proposed generalized FE model was performed to demonstrate the efficiency of technique in modeling of arbitrary interfaces in large deformations by several numerical examples. Numerical simulations of problems with rewrelatively complex geometry were presented. The examples were solved using both the G-FEM and FEM techniques and the results were compared. The numerical results clearly demonstrate the capability of proposed technique in modeling large elastic deformations with multiple material interfaces.

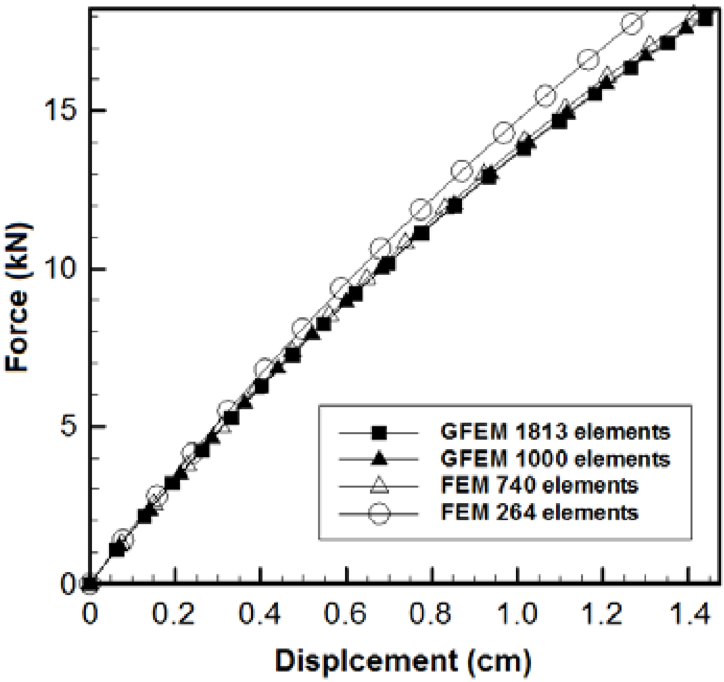


Figure 11. The variations of reaction force with horizontal displacement; A comparison between the FEM and generalized–FEM techniques

REFERENCES

[1] Belytschko T, Lu Y, Gu L. Element-free Galerkin methods. *Computer Methods in Applied Mechanics and Engineering.* 1994; **37**: 229–256.

[2] Belytschko T, Krongauz Y, Organ D, Fleming M, Krysl P. Mesh less methods: an overview and recent developments. *Computer Methods in Applied Mechanics and Engineering*. 1996; **139**: 3–47.

[3] Fleming M, Chu YA, Moran B, Belytschko T. Enriched element free Galerkin methods for crack tip fields. *International Journal for Numerical Methods in Engineering*. 1997; **40**: 1483–1504.

[4] Rashid M. The arbitrary local mesh refinement method: an alternative to remeshing for crack propagation analysis. *Computer Methods in Applied Mechanics and Engineering.* 1998; **154**: 133–150.

[5] Oliver J. Continuum modeling of strong discontinuities in solid mechanics using damage models. *Computational Mechanics*. 1995; **17**: 49–61.

[6] Strouboulis T, Copps K, Babuska I. The Generalized Finite element method, *Computer Methods in Applied Mechanics and Engineering*. 2001; **190**: 4081–4193.

[7] Duarte CA, Hamzeh ON, Liszka TJ, Tworzydlo WW. A generalized finite element method for the simulation of three-dimensional dynamic crack propagation, *Computer Methods in Applied Mechanics and Engineering*. 2001; **190**: 2227–2262.

[8] Belytschko T, Black T. Elastic crack growth in finite elements with minimal remeshing, *International Journal for Numerical Methods in Engineering*. 1999; **45**: 601–620.

[9] Daux C, Moës N, Dolbow J, Sukumar N, Belytschko T. Arbitrary branched and intersecting cracks with the extended finite element method, *International Journal for Numerical Methods in Engineering*. 2000; **48**: 1741–1760.

[10] Belytschko T, Moës N, Usui S, Parimi C. Arbitrary discontinuities in finite elements, *International Journal for Numerical Methods in Engineering*. 2001; **50**: 993–1013.

[11] Moës N, Belytschko T. Extended finite element method for cohesive crack growth, Engineering *Fracture Mechanics*. 2002; **69**: 813–833.

[12] Ventura G, Budyn E, Belytschko T. Vector level sets for description of propagating cracks in finite elements. *International Journal for Numerical Methods in Engineering*. 2003; **58**: 1571–1592.

[13] Lee SH, Song JH, Yoon YC, Zi G, Belytschko T. Combined extended and superimposed finite element method for cracks. *International Journal for Numerical Methods in Engineering*. 2004; **59**: 1119–1136.

[14] Areias PMA, Belytschko T. Analysis of three-dimensional crack initiation and propagation using the extended finite element method. International *Journal for Numerical Methods in Engineering*. 2005; **63**: 760–788.

[15] Khoei AR, Shamloo A, Azami AR. Extended finite element method in plasticity forming of powder compaction with contact friction. *International Journal of Solids and Structures*. 2006; *43*: 5421–5448.

[16] Khoei AR, Biabanaki SOR, Anahid M. Extended finite element method for three-dimensional large plasticity deformations. *Computer Methods in Applied Mechanics and Engineering*. 2008; **197**: 1100–1114.

[17] Khoei AR, Anahid M, Shahim K. An extended arbitrary Lagrangian–Eulerian finite element method for large deformation of solid mechanics. *Finite Elements in Analysis and Design*. 2008; **44**: 401–416.

[18] Khoei AR, Nikbakht M. An enriched finite element algorithm for numerical computation of contact friction problems. *International Journal of Mechanical Sciences*. 2007; **49**: 183–*199*.

[19] Khoei AR, Biabanaki SOR, Anahid M. A Lagrangian – extended finite element method in modeling large plasticity deformations and contact problems. International Journal of *Mechanical Sciences*. 2009; **51**: 384–401.

[20] Khoei AR, Taheri-Mousavi SM. Modeling of large deformation – large sliding contact via the penalty X-FEM technique. *Computational Materials Science*. 2010; doi:10.1016/j.*commatsci*.2010.02.008, in press.

[21] Ventura G, Moran B, Belytschko T. Dislocations by partition of unity. *International Journal for Numerical Methods in Engineering.* 2005; **62**: 1463–1487.

[22] Fries TP, Belytschko T. The intrinsic XFEM: a method for arbitrary discontinuities without additional unknowns. *International Journal for Numerical Methods in Engineering.* 2006; **68**: 1358–1385.
[23] Gracie R, Ventura G, Belytschko T. A new fast finite element method for dislocations based on interior discontinuities. *International Journal for Numerical Methods in Engineering*. 2007; **69**: 423–441.
[24] Ventura G. On the elimination of quadrature subcells for discontinuous functions in the extended finite-element method. *International Journal for Numerical Methods in Engineering*. 2006; **66**: 761–795.
[25] Holdych DJ, Noble DR, Secor RB. Quadrature rules for triangular and tetrahedral elements with generalized functions. *International Journal for Numerical Methods in Engineering*. *2008*; **73**: 1310–1327.
[26] Li Z, Lin T, Wu X. New Cartesian grid methods for interface problems using the finite element *formulation*. *Numerische Mathematik* 2003; 96:61–98.
[27] Sukumar N, Tabarraei A. Conforming polygonal finite elements. *International Journal for Numerical Methods in Engineering*. 2004; **61**: 2045–2066.
[28] Noble DR, Newren EP, Lechman JB. A conformal decomposition finite element method for modeling stationary fluid interface problems. *International Journal for Numerical Methods in Fluids*. 2010; **63**: 725–742.
[29] Wachspress EL. *A Rational Finite Element Basis*. Academic Press, New York. 1975.
[30] Coxeter HSM. *Introduction to Geometry.* Wiley, New York. 1961.
[31] Gout JL. Rational Wachspress-type finite elements on regular hexagons. *IMA Journal of Numerical Analysis*. 1985; **5**: 59 –77.
[32] Warren J. Barycentric coordinates for convex polytopes. *Advances in Computational Mathematics.* 1996; **6**: 97 –108.
[33] Dahmen W, Dikshit HP, Ojha A. On Wachspress quadrilateral patches. *Computer Aided Geometric Design*. 2000; **17**: 879–890.
[34] Dasgupta G. Interpolants within convex polygons: Wachspress' shape functions. *Journal of Aerospace Engineering.* 2003; **16**: 1– 8.
[35] Dasgupta G. Integration within polygonal finite elements. *Journal of Aerospace Engineering*. 2003; **16**: 9–18.
[36] Rashid MM, Gullett PM. On a finite element method with variable element topology. *Computer Methods in Applied Mechanics and Engineering*. 2000; **190**: 1509 –1527.
[37] Meyer M, Lee H, Barr A, Desbrun M. Generalized barycentric coordinates for irregular polygons. *Journal of Graphics Tools*. 2002; **7**: 13–22.